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Mathematical Achievement and the Specific Problem

The study of mathematics for discipline or culture or information or possible later application motivates but slightly the mental powers of the average student. It is the *problem*, with its challenge to these powers, that is most likely to inspire him to whole-hearted effort.

Archimedes made his discoveries by running down the solutions of previously unsolved mathematical or mechanical problems. Devoted as he was to pure mathematics, Fermat schieved his successes in the theory of numbers because of the maximum power inspired by his desire to solve some specific problem. His discoveries in the differential calculus came out of efforts to find the meaning of continuity in a curve.

Newton, ranked alongside of Archimedes, and, along with Leibniz discoverer of the modern calculus, created most of his mathematics in his colossal task of solving the *problem* of universal gravitation. Evariste Galois' supreme problem was that of finding the conditions for the solvability of the general equation. He solved this problem before he reached his twentieth year and wrote it out shortly before he was killed in a duel.

In this manner could be cited almost endless examples of *problem-inspired* achievements in mathematics. Vast as is the body of mathematical science today, it yet stands silent witness to the superior influence of the unresolved question in mathematics in bringing to fullest expression the athletic instincts of strong and average minds alike.

S. T. Sanders.

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A Representation of Certain Integer Powers

By E. T. Bell
California Institute of Technology

1. The ternary form $F_{r,t}(x,y,z)$, where r,t are non-negative ingegers, is defined by

$$F_{r,t}(x,y,z) \equiv x^r y^t + y^r z^t + z^r x^t.$$

Theorem 1. *If n,a are positive integers, and s is the L. M. C. of $3,a+2,2a+1$, positive integers ξ,η,ζ may be found such that*

$$n^s = F_{2a+1,a+2}(\xi,\eta,\zeta)/F_{a-1,2a+1}(\xi,\eta,\zeta). \tag{1}$$

It will suffice to exhibit ξ,η,ζ. Write

$$s/3 \equiv f, \quad s/(a+2) \equiv g, \quad s/(2a+1) \equiv h.$$

Then the values

$$b^f c^g d^{ah}, \quad b^f d^{(a+1)h}, \quad b^f c^{(a+1)g}$$

of ξ,η,ζ respectively reduce the right of (1) to $(bcd)^s$. Positive integers b,c,d may be chosen such that $n=bcd$. If

$$n = \prod_{i=1}^{u} p_i^{a_i}$$

is the prime-factor decomposition of n, the number of sets (b,c,d) is

$$2^{-u} \prod_{i=1}^{u} (a_i+1)(a_i+2). \tag{2}$$

For $a>1$, Theorem 1 is a special case of Theorem 3 proved in §3.

2. If a,b are coprime positive integers, $N \equiv N(a,b)$ denotes the total number of integer parameters necessary and sufficient for the complete integer solution x,y,z of

$$x^{a+b} y^a = z^{2a+b}. \tag{3}$$

When a,b are given, $N(a,b)$ is uniquely determined and may be found nontentatively.

The complete integer solution of (3) is of the form

$$x = \prod \phi_i^{x_i}, \quad y = \prod \phi_i^{y_i}, \quad z = \prod \phi_i^{z_i}, \tag{4}$$

the product extending to $i=1,\cdots,N$, where the ϕ_i are N independent integer parameters, the x_i,y_i,z_i are non-negative integers, and

$$\sum x_i\neq 0, \quad \sum y_i\neq 0, \quad \sum z_i\neq 0.$$

For positive values of the ϕ_i, all of x,y,z and $F_{r,t}(x,y,z)$ are positive integers, and for such solutions x,y,z of (3) the product

$$(x^{a+b}y^a-z^{2a+b})(y^{a+b}z^a-x^{2a+b})(z^{a+b}x^a-y^{2a+b})$$

vanishes. Whence, for such x,y,z,

$$(xyz)^a=F_{3a+2b,3a+b}(x,y,z)/F_{b,3a+2b}(x,y,z). \tag{5}$$

With $s_i\equiv x_i+y_i+z_i$, let s be the L. M. C. of the $s_i, i=1,\cdots,N$, and write $t_i\equiv s/s_i$. Let the n_i be any N positive integers whose product is n, and take $\phi_i\equiv n_i^{t_i}$ in (5). Then, from (4), it follows that

$$xyz=n^s, \quad n=\prod n_i. \tag{6}$$

Theorem 2. *If*

$$n=\prod_{i=1}^{N} n_i$$

any decomposition of the positive integer n into a product of N positive integers n_i, where N is as defined for (3), *x_i,y_i,z_i are as in* (4), *and s,t_i are as defined after* (5), *then*

$$n^{as}=F_{3a+2b,3a+b}(\xi,\eta,\zeta)/F_{b,3a+2b}(\xi,\eta,\zeta),$$

$$\xi\equiv\prod n_i^{x_it_i}, \quad \eta\equiv\prod n_i^{y_it_i}, \quad \zeta\equiv\prod n_i^{z_it_i}.$$

This is immediate from (5), (6). With the a_i as in (2), the number of decompositions of n is

$$[(N-1)!]^{-u}\prod_{i=1}^{u}(a+1)\cdots(a_i+N-1), \tag{7}$$

and each decomposition gives a set of values of ξ,η,ζ representing n^{as} in the form

$$F_{3a+2b,3a+b}(x,y,z)/F_{b,3a+2b}(x,y,z). \tag{8}$$

In addition to the representation $(x,y,z)=(\xi,\eta,\zeta)$, there are the two obtained from this by cyclic permutations of ξ,η,ζ, namely, $(x,y,z)=(\eta,\zeta,\xi)$, (ζ,ξ,η).

Theorem 3. *If a,b are coprime positive integers, and n is a positive integer, a positive integer s and positive integers ξ,η,ζ may be found such that $x=\xi$, $y=\eta$, $z=\zeta$ represent n^{as} in the form* (8).

From Theorem 2. The proof of (7) follows at once on comparing the expansion of the Nth power of Riemann's zeta function with the distributed form of the equivalent Euler product.

Tensor Algebra and Invarints, II

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(Concluded from the October, 1944, Issue)

7. *Numerical tensors.* A tensor whose components are constants in all coordinate systems is called a numerical tensor. There is no numerical tensor of the first order under unrestricted transformations, except the degenerate one with every component zero. Consider $t_i{}^i T_j{}^k \bar{\delta}_k{}^l$. Now $T_j{}^k \delta_k{}^l = T_j{}^l$ and $t_i{}^i T_j{}^l = \delta_j{}^i$. That is $\delta_j{}^i = t^{li} T_j{}^k \bar{\delta}_k{}^l$. So $\delta_j{}^i$ is an absolute numerical tensor of order two. From the results of Cramlet [3] and Knebelman [4], except for a scalar multiplier, we know that $\delta_j{}^i$ is the only absolute numerical tensor of order 2. Further, it is shown in these papers that only a mixed tensor having equal contravariant and covariant order can be an absolute numerical tensor with respect to a unrestricted transformation. The expressions

$$\delta^{ij}_{kl} = \begin{vmatrix} \delta_k{}^i & \delta_l{}^i \\ \delta_k{}^j & \delta_l{}^j \end{vmatrix} \quad \text{and} \quad \delta^{ijk}_{lmn} = \begin{vmatrix} \delta_l{}^i & \delta_m{}^i & \delta_n{}^i \\ \delta_l{}^j & \delta_m{}^j & \delta_n{}^j \\ \delta_l{}^k & \delta_m{}^k & \delta_n{}^k \end{vmatrix}$$

are also absolute numerical tensors, since each is the sum of absolute numerical tensors of the same order. More generally, $\delta^{i_1 \cdots i_p}_{j_1 \cdots j_p}$ is an absolute numerical tensor, where

$$\delta^{i_1 \cdots i_p}_{j_1 \cdots j_p} = \sum \delta_{j_1}{}^{i_1} \cdots \delta_{j_p}{}^{i_p} \qquad (p \leq n) \tag{18}$$

in which Σ indicates that the sum is to be formed of all products of the $\delta_{j_1}{}^{i_1}$, $\delta_{j_2}{}^{i_2}, \cdots, \delta_{j_p}{}^{i_p}$ that can be formed by permutation of the $i_1, \cdots, i_p$ when each product has a $\pm$ sign prefixed according as the permutation is even or odd. The expression (18) is called the generalized Kronecker delta, or the determinant tensor. Murnaghan gave some properties of it [5] and later established its tensor character [6]. It was established by Cramlet [3] and Knebelman [4] that the most general absolute numerical tensor is the function defined by

$$\gamma^{i_1 \cdots i_p}_{j_1 \cdots j_p} = \sum A_p \, \delta_{j_1}{}^{i_1} \cdots \delta_{j_p}{}^{i_p} \tag{19}$$

where the Σ indicates that the set $i_1, \cdots, i_p$ is permuted in all possible ways and the $p!$ terms so formed, each multiplied by an arbitrary scalar, are added. When the scalers A_p are all equal to $+1$ we have what is commonly called the permanent tensor, which is denoted by $\pi^{i_1 \cdots i_p}_{j_1 \cdots j_p}$ [7].

A specialization, by means of characters of the symmetric group, of the general numerical tensor (19), gives [8] for each partition of p an important tensor ${}_\alpha I^{i_1\cdots i_p}_{j_1\cdots j_p}$, called an immanant tensor. For a given value of p these are mutually orthogonal idempotents, in the sense that

$$ {}_\alpha I^{i_1\cdots i_p}_{l_1\cdots l_p} \cdot {}_\alpha I^{l_1\cdots l_p}_{j_1\cdots j_p} = {}_\alpha I^{i_1\cdots i_p}_{j_1\cdots j_p}; \quad {}_\alpha I^{i_1\cdots i_p}_{l_1\cdots l_p} \cdot {}_\beta I^{l_1\cdots l_p}_{j_1\cdots j_p} = 0. $$

Here, as elsewhere, we represent the null tensor by 0. By multiplying and contracting an arbitrary tensor $T_{i_1\cdots i_p}$ with an immanent, a tensor $I^{l_1\cdots l_p}_{i_1\cdots i_p} T_{l_1\cdots l_p}$ is obtained, which belongs to a particular type of symmetry associated with the partition $[\alpha]$ of p Thus for $p=2$ there are two immanents, the permanent $\frac{1}{2}\pi_{j_1}{}^{i_1}{}_{j_2}{}^{i_2}$, and the determinant $\frac{1}{2}\delta_{j_1}{}^{i_1}{}_{j_2}{}^{i_2}$. Now

$$ \tfrac{1}{2}\pi_{j_1}{}^{i_1}{}_{j_2}{}^{i_2} T_{i_1 i_2} = \tfrac{1}{2}(T_{j_1 j_2} + T_{j_2 j_1}), \quad \tfrac{1}{2}\delta_{j_1}{}^{i_1}{}_{j_2}{}^{i_2} T_{i_1 i_2} = \tfrac{1}{2}(T_{j_1 j_2} - T_{j_2 j_1}). $$

The first of these tensors is symmetric in the indices $j_1 j_2$, and the second skew-symmetric; further $T_{i_1 i_2}$ is the sum of these two specialized tensors. Only when $p=2$ is it sufficient that the skew-symmetric part of a tensor be zero in order that the tensor be symmetric. The expression ${}_\alpha I^{i_1\cdots i_p}_{i_1\cdots i_p}$ is an important numerical invariant, it being the number of independent components in a tensor of the corresponding type of symmetry [9].

Associated with every coordinate system of n varaibles are two relative, numerical, skew-symmetric tensors of the nth order, $\epsilon^{i_1\cdots i_n}$, a contravariant tensor of weight one, and $\epsilon_{i_1\cdots i_n}$, a covariant tensor of weight minus one, such that every other relative numerical tensor under an unrestricted transformation is a polynomial in terms of these [10], [11]. These relative numerical tensors occur frequently in geometry. Craig has shown [12] how the relative and absolute numerical tensors may be used in establishing identities in vector analysis. For $n=3$, ϵ^{ijk} and ϵ_{ijk} may be defined by the relations

$$ \epsilon^{ijk} = \delta^{ijk}_{123} = \begin{vmatrix} \delta_1{}^i & \delta_2{}^i & \delta_3{}^i \\ \delta_1{}^j & \delta_2{}^j & \delta_3{}^j \\ \delta_1{}^k & \delta_2{}^k & \delta_3{}^k \end{vmatrix}, \qquad \epsilon_{lmn} = \delta^{123}_{lmn} = \begin{vmatrix} \delta_l{}^1 & \delta_m{}^1 & \delta_n{}^1 \\ \delta_l{}^2 & \delta_m{}^2 & \delta_n{}^2 \\ \delta_l{}^3 & \delta_m{}^3 & \delta_n{}^3 \end{vmatrix}, $$

and have the values: 0, when two indices are equal; 1, when the indices are arranged as an even permutation of 1,2,3; -1, when the indices are arranged as an odd permutation of 1,2,3. The tensor ϵ^{ijk} can be represented explicitly by sectioning it into the three non-tensorial 2-matrices

$$ \epsilon^{\overline{ij}1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \epsilon^{\overline{ij}2} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \epsilon^{\overline{ij}3} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

Its law of transformation is

$$\epsilon^{ijk} = |t|^{-1} t_s{}^i t_t{}^j t_u{}^k \epsilon^{-stu}, \quad \text{or} \quad \epsilon^{-stu} = |t|^{+1} T_i{}^s T_j{}^t T_k{}^u \epsilon^{ijk}.$$

8. *Construction of projective concomitants.* In Section 2 we saw that B_i in the form $B_i x^i$ is a covariant tensor of order one. In general it can be shown that $B_{i_1 \cdots i_p}$ in $B_{i_1 \cdots i_p} x^{i_1} \cdots x^{i_p}$ is a covariant tensor of order p, and that $C^{i_1 \cdots i_p}$ in $C^{i_1 \cdots i_p} u_{i_1} \ldots u_{i_p}$, where $x^i u_i = 0$, is a contravariant tensor of order p. Such tensors will be referred to as the coefficient tensors of these respective ground forms. Cramlet [11] laid the basis for the construction, with the use of tensor algebra, of algebraic invariants under unrestricted transformations. His result may be stated in the form of

Theorem I. *Every projective concomitant for a set of ground forms in x^i and u_i is expressible by composition as a tensor of order zero with the use of the coefficient tensors of the ground forms and the tensors $\epsilon^{i_1 \cdots i_n}$, $\epsilon_{i_1 \cdots i_n}$, x^i, and u_i.*

Example 1. The discriminant D of the ternary quadratic $B_{ij} x^i x^j$ is given by $6D = B_{ij} B_{kl} B_{mn} \epsilon^{ikm} \epsilon^{jln}$.

Example 2. The line equation of the conic $B_{ij} x^i x^j = 0$ is $B_{ij} B_{kl} \epsilon^{ikm} \epsilon^{jln} U_m U_n = 0$.

9. *Geometries as the theories of tensors.* Klein's viewpoint of a geometry as the invariant theory of a transformation group, as formulated in the Erlanger Program [13] in 1870, has played an important part in the study of geometry during the past half century. In the last two decades the viewpoint of a geometry as the theory of a tensor has received considerable theoretical discussion and utilization in connection with the new differential geometries [14]. But there are advantages, particularly from the invariant theoretic point of view, of regarding a subgeometry of projective geometry as the theory of an adjoined numerical tensor [15]. From this point of view we regard the study of geometry as the study of those configurations which remain invariant with respect to a certain type of transformation.

Affiine geometry. If in the $(n-1)$ projective geometry we hold the linear form $L_i x^i$ latent, where $L_i = (0, \cdots, 0, 1)$, we have the special numerical tensor L_i. Then from Theorem I we have

Theorem II. *Affine geometry as a subgeometry of projective is the theory of the numerical tensor L_i.*

This means that knowing the projective invariants for a set of forms, as given by Theorem I, we can construct all the affine invariants

for this set of forms by adding to our list of basic tensors L_i. In applications it is advantageous to introduce the tensor $A^{i_1\cdots i_{n-1}}$ where $A^{i_1\cdots i_{n-1}}=\epsilon^{i_1\cdots i_n}L_n$. For $n=3$,

$$A^{ij}=\epsilon^{ijk}L_k=\epsilon^{ij3}=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}.$$

It should be noted that while this matrix is not a tensor under the unrestricted projective transformation, it is a tensor under the restricted transformation which leaves the line at infinity fixed. Combining Theorems I and II we have

Theorem III. *Every affine concomitant for a set of ground forms in x^i and u_i is expressible by composition as a tensor of order zero with the use of the coefficiant tensors of the ground forms and the tensors $\epsilon^{i_1\cdots i_n}$, $\epsilon_{i_1\cdots i_n}$, L_i, $A^{i_1\cdots i_{n-1}}$, x^i, and u_i.*

Example 3. The lines $(n=3)$ $B_ix^i=0$ and $C_ix^i=0$ have the affine invariant $B_iC_jA^{ij}$. If this invariant is zero, the lines are parallel.

Example 4. The conic $B_{ij}x^ix^j=0$ $(n=3)$ is a parabola if $B_{ij}B_{kl}A^{ik}A^{jl}$ is zero. The equation of the center of this conic is $B_{ij}B_{kl}\epsilon^{ikm}A^{jl}u_m=0$.

Example 5. The line $B_ix^i=0$ and the conic $C_{ij}x^ix^j=0$ $(n=3)$ have the affine invariant $C_{ij}A^{ik}A^{jl}B_kB_l$.

Euclidean (parabolic metric) geometry. Holding the quadratic form $E^{ij}u_iu_j=0$ latent, where

$$E^{ij}=\begin{bmatrix}1&0&0&\cdots&0&0\\0&1&0&\cdots&0&0\\\cdot&\cdot&\cdot&\cdots&\cdot&\cdot\\0&0&0&\cdots&1&0\\0&0&0&\cdots&0&0\end{bmatrix},$$ we have

Theorem IV. *Euclidean geometry as a subgeometry of affine geometry is the theory of the tensor E^{ij}, and as a subgeometry of projective geometry is the theory of the tensors L_i and E^{ij}.*

Theorem V. *Every Euclidean concomitant for a set of ground forms in x^i and u_i is expressible by composition as a tensor of order zero with the use of the coefficient tensors of the ground forms and the tensors $\epsilon^{i_1\cdots i_n}$, $\epsilon_{i_1\cdots i_n}$, L_i, $A^{i_1\cdots i_{n-1}}$, E^{ij}, x^i, and u_i.*

Example 6. The line $(n=3)$ $B_ix^i=0$ has the euclidean invariant $B_iB_jE^{ij}$. For $n=4$ the plane $B_ix^i=0$ has a euclidean invariant of the same form.

Example 7. The conic $(n=3)$ $C_{ij}x^ix^j=0$ has the euclidean invariants $C_{ij}E^{ij}$, $C_{ij}C_{kl}A^{ik}A^{jl}$, $C_{ij}C_{kl}C^{mn}\epsilon^{ikm}\epsilon^{jln}$.

Theorem V has been used in a detailed study of the euclidean concomitants of the ternary cubic [16], and of three lines [17].

Elliptic metric geometry. We shall close with a brief discussion of how a metric non-euclidean geometry [18] may be treated from the tensorial view-point. In the projective plane let the line conic $\mathcal{E}^{ij}u_iu_j=0$ be latent, where

$$\mathcal{E}^{ij}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

The body of definitions and theorems which express properties that are left unchanged under the latency of this conic is called elliptic metric geometry.

Theorem VI. *Elliptic metric geometry as a subgeometry of projective geometry is the theory of the tensor* $\mathcal{E}^{ij}$.

Theorem VII. *Every elliptic concomitant for a set of ground forms* $(n=3)$ *in* x^i *and* u_i *is expressible by composition as a tensor of order zero with the use of the coefficient tensors and the tensors* ϵ^{ijk}, ϵ_{ijk}, $\mathcal{E}^{ij}$, x^i, and u_i.

Example 8. The conic $B_{ij}x^ix^j=0$ has the elliptic invariants $B_{ij}\mathcal{E}^{ij}$, $B_{ik}B_{jl}\mathcal{E}^{il}\mathcal{E}^{jk}$, and $B_{ij}B_{kl}B_{mn}\epsilon^{ikm}\epsilon^{jln}$.

The theorems given here constitute a basis for the construction of algebraic invariants. The problem of the reduction of such invariants can likewise be handled with the tensor algebra, the tool being an appropriate set of tensor identities. A complete set of tensor identities for the reduction of euclidean concomitants of ternary forms in x^j is established and used in [16]

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Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

Pertinent Historical Material for a Slide Rule Course

By NORMA SLEIGHT
New Trier Township High School
Winnetka, Illinois

A credit course in the theory and use of the slide rule was given for the first time this past year in New Trier Township High School, Winnetka, Illinois. Starting as an after school class for military minded boys five years ago, the emphasis and personnel of the groups has changed radically. Nearly half of the enrollment now is composed of girls and attention is directed toward applications and uses of the rule in the fields of science, mathematics, vocations, and industry. Since a knowledge of logarithms is a prerequisite, the classes are above average in ability and interest.

The scales studied are the *C*, *D*, *A*, *B*, *K*, *S*, and *T*, the more difficult types of problems being of this nature:

$$V = 8.3\sqrt{\frac{12\times 2370}{.47\times 198}}.$$

While a certain degree of mastery of techniques is necessary and desirable, appreciation for the development and use of the slide rule is emphasized. A study of the historical background of most phases of science or mathematics places a student in a more sympathetic and receptive state of mind. Even though many of the students may never use the slide rule to any great extent, the intelligent person likes to know what is going on about him and each contact with the history of our culture will make his general reading more interesting during his adult days.

Since the historical background material for the purpose was found to be very scattered and perhaps too difficult, the following was collected and assigned as a part of the work of the course.

There is a tendency for people of any time and locality to assume that customs and material things as experienced in their lives are static; they have been that way for a long time and will remain so far many years to come. We, at present, here in the United States accept porcelain bath-tubs, decimals, radios, congress, comptometers, and check books as though they had always existed. Everything on this list, however, is of comparatively recent origin. Columbus was not familiar with one item here mentioned.

The purpose of this account is to review briefly a history of methods of counting, the evolution of our present number system, and to list some of the outstanding calculating devices of the past and present. Our current methods of representing and operating with numbers (natural, vulgar fractions, and decimal fractions) have been "handed to us on a platter." A brief bout with the past makes us realize two things: one, that because of great improvement in both symbolism and operational methods, the average high school student can be more glib with figures than was Archimedes; the other idea which comes as a shock is that notation and computation have been in a constant state of change throughout recorded history. Therefore it is practical to assume that our present system is not crystallized but is still in a state of flux. Perhaps some reader of this material will be encourated to study more systematically the records of the past and thus find a way to improve arithmetic as it now exists.

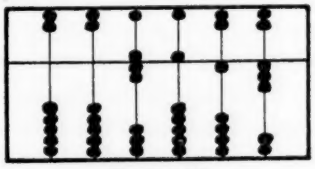
Let us turn back a few centuries. Even though arithmetic is as fundamental to mathematics and science as the alphabet is to literature; we cannot trace each aspect of the subject to its source. Counting was probably practiced before the advent of language, it having been accomplished by notching sticks—thus the word *tally*. In Roman times calculation was performed with shells and pebbles from the beaches—thus the word calculus, meaning pebble. Herodotus (B.C. 440) records this custom of counting among civilized peoples. Think of the ancient astronomers of Egypt turning to their pebble boxes to carry on their work with the calendar and eclipse forecasting! "We may picture the equipment of the students of the University of Alexandria to have consisted of a sand tray and a pointer for the diagrams, and a box or bag of pebbles for the calculations, as they studied those treatises of the famous professor which we call Euclid."[1] Their arithmetical results were achieved in a clumsy way. In Shakepeare's time tradesmen were still casting accounts on the "counter". We note the derivation of our present term, counter, and the expression "over the counter". In spite of all our modern devices of record keeping, there are a few people, even in the United States, who employ

[1] Cunningham, *The Story of Arithmetic*, p. 12.

this ancient and medieval system of counters in their reckoning; note the poker players with their chips.

The natural outcome from the pebble counting was the abacus, a neat device for keeping pebbles or counters in line. (See Figure 1). Since the numerals of the ancients were unsuited to calculations, and as numbers were growing larger, some system of mechanical computation had to be devised.

Figure 1



The early abacus probably was a sand table equipped with stylus, later a ruled table on which counters could be arranged. In some places it appeared in a different form,—grooves or rods on which movable disks were placed. These are still found in use in China, Russia, Japan, and Chicago's Chinatown. Note that, because the position of a counter indicated its potency, units, tens, hundreds, etc., no zero was necessary. And since ten units accumulated on one bar are replaced by one bead on the bar directly to the left, we may have the origin of our expression in addition, "to carry".

The change from the habit of counting with material things to that of representation' of numbers by symbols was a great step. We are all familiar with the Roman numerals (letters, actually). They were very clumsy and certainly didn't invite advance in the methods of computation. Imagine trying to multiply 356 by 47, using Roman numerals!

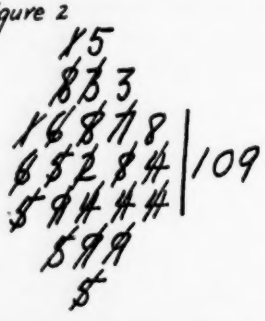
The time and place of the origin of our present Hindu-Arabic system of notation are shrouded in uncertainty. Sporadic evidences have been discovered from time to time but findings continually change the conclusions. We do suspect that our present system may have originated in India; it certainly was carried to Bagdad in the 8th century, then to Europe. The numerals appeared in Alexandria in the 5th century by way of the Eastern trade routes, and in Spain during the 10th century. Some authors claim the symbols were used in India the 3rd century B. C. but their use in computation was nil until the invention of zero, giving the digits place value. "The earliest undoubted occurrence of a zero in India is seen in an inscription of 876 at Gwalior. In this inscription 50 and 270 are both written with zeros. We have the information, however, that a place value was

recognized at an earlier period, so that the zero had probably been known for a long time."[1] Evidence is coming to light now that the Chinese used characters similar to our numbers 2000 B.C. in a well established system of bookkeeping and had a symbol for zero.

Adoption of the Arabic figures in Medieval calendars did more than anything else to popularize the Arabic characters but the Florentine bankers of the 13th century were forbidden to use them. "But no inscription in England has been found with the date written in the new symbols belonging to an earlier date than the 16th century."[2] From then on their use spread widely and we find ourselves possessors of the present number symbols where there are two values for each digit, one is intrinsic, determined by shape, and the other is local, depending upon position. Furthermore, *10* is the radix of our scale,[3] that is, each unit in a given column indicates ten times the value of the same unit in a column immediately to the right.

Speaking of the fundamental operations, it is impossible to trace back to the origin of addition and subtraction these being very simple to perform with small numbers; but multiplication and division, more difficult and less needed, came later. Robert Recorde, an English writer of the middle of the 16th century, tells us how to multiply and divide on the abacus. An old method of division called the galley method was practiced in Europe until 1600. (See Figure 2). This

Figure 2



probably came from the Hindus and earned its name from the fact that the number layout resembled the old galley ships. Here is a sample, showing 65284 divided by 594. The result is 109 with a remainder of 538. Long division as we know it first appeared in print in 1491.[5] The first practical arithmetic to appear in print bears the

[1] Smith, D. E., *History of Mathematics*, Vol. II, p. 69.

[2] Cunningham, *ibid.*, p. 42.

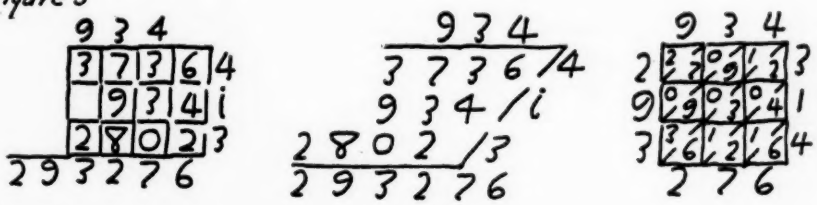
[3] The radix can be any number. 12, 20, and 60 have been used.

[4] Smith, D. E., *Ibid.*, Vol. II, p. 137.

[5] Smith, D. E., *Ibid.*, Vol. I, p. 255. From Filippo Calamdri's Arithemtic, Florence, 1941.

date 1478 but is anonymous, called the Treviso arithmetic. It presents long division in the galley form but multiplication in our present form. Here are the other layouts for multiplication as given in this book:

Figure 3



The symbols, plus and minus, as we know them were used in the Italian warehouses of the 15th century but denoted excess and deficiency. The 16th century saw the advent of their present meaning, and times, " × ", was first used by Oughtred in "Clavis Mathematica".

The common fraction has always been a contrivance of which we stand in awe. In modern phraseology, the human race has been alergic to fractions since the beginning of their use. The English, rather than to speak of $\frac{1}{4}$ of a pound, divide it into smaller parts and call it 4 whole ounces! This is the story book of many a small unit of measurement. However, representations of fractions (not operations) date back at least to the 3rd Dynasty of Ur (2400 B.C.). One of the books of Ahmes (1550 B.C.) lists fractions. Bhaskara's arithmetic (12th century) has a whole section on fractions and Pythagoras couldn't state his laws of vibrating strings without using them. He was the gentleman who said, "Where there is harmony, there are numbers." Operations with fractions are of more recent date. In Queen Elizabeth's time arithmetic was taught with the aid of jingles, the emphasis was on memory rather than understanding. Sounds like the era of the "Little Red School House" and Ichabod Crane, doesn't it? Here is one rule learned by our English cousins of old:

> "Addition of fractions and likewise subtraction
> Requireth that first they all have like bass-es
> Which, by reduction, is brought to perfection,
> And being once done, as ought in like cases,
> Then add or subtract their tops and no more,
> Subscribing the bass-es made common before."[2]

We have evidence of multiplication of fractions in the 15th century[3] and division was in general use in the 17th century.[4]

[1] Smith, D. E., Rara Arithmetica, p. 5. From The Treviso Arithmetic, 1478.

[2] Cunningham, *Ibid.*, p. 149.

[3] Calamdri's Arithmetic, 1491.

[4] Smith, *Ibid.*, Vol. II, p. 227. Known four or five hundred years earlier by Hindu and Arab writers.

Decimals. What a wonderful invention for the practical scientist who must measure and compute! Since this is one of the most useful developments of arithmetic, let us observe the evolution of notation:

Figure 4

Dates	Notations
1585	23 5'4"9'''6""
1590	23⓪5①4②9③6④
1610	23⑴5⑵4⑶9⑷6⑸
1619	23 5496
1631	23 |5496
1635	23|5496
1651	23.5496

Our first reaction is "How could the great minds of the past be so simple?" Well, how could they? Perhaps a hundred years from now, people will say the same of our mathematicians.

We are not certain just from whence came our decimal fraction but the point appeared the first time in print in 1492, in Pellizzati's arithmetic. Even as late as 1816 in New York, Pike's arithmetic used the bar for the decimal point and the English texts of the early part of this century didn't show the decimal fraction in our present form. While the notation may not be settled even yet, the theory and uses are. The invention of logarithms (1614) had more to do with the development and use of the decimal fraction than any other single influence.

Just why were logarithms invented at this time? Through the Dark Ages and Medieval times the discoveries of science were patiently copied in the monasteries where these copies rested undisturbed and with few eyes to behold them. The invention of printing (15th century) and the surge of intellectual activity exhibited by the Renaissance had a profound effect upon the development of science and mathematics. Until this period, small numbers sufficed.

The word million first appeared in printed form in the Treviso arithmetic. Million was a luxury word then and even the great scholars called it a thousand thousand. The Ancient Egyptians represented a million by a picture of a man holding up his hands in astonishment. Until the world war of 1914-1918 taught the world to think in billions there was not much need for number names beyond millions.[2] Numbers could be expressed in figures, and an astronomer

[1] Cunningham, *Ibid.*, p. 139.
[2] Smith, *Ibid.*, Vol. II, p. 84.

could write a number like 9.15×10^{7} or 2.5×10^{20} without caring about a name.[1] At present we speak glibly of billions.

Let us glance at a few outstanding events, together with the imposing list of famous scientists and other great men of the 16th and 17th centuries, remembering that the first treatise on logarithms was published by Napier in 1614 in Edinburgh.

1450 (about)	Printing from movable type in Europe.
1452-1519	Leonardo daVinci—optics, geometry, art, mechanics.
1492	Columbus discovered America.
1473-1543	Nicholas Copernicus—astronomy, trigonometry.
1475-1564	Michelangelo—architecture, military engineering, sculpture, art.
1564-1642	Galileo Galilei—geometry, astronomy, mechanics.
1642-1727	Isaac Newton—calculus, binomial expansion, light, universal gravitation, theory of equations.
1506-1557	Nicola Tartaglia—cubic equation, general mathematics, commercial arithmetic.
1501-1576	Girolamo Cardano—cubic equation, gambling, algebraist.
1541	DeSoto finds the Mississippi.
1548-1620	Simon Stevin—theory of decimals, statics, hydrostatics (from the Low Countries).
1620	Landing of the Pilgrims.
1564-1616	Shakespeare.
1571-1630	John Kepler—astronomy, geometry.
1522-1560	Lodovico Ferrari—4th degree equation.
1540-1603	Francois Viete—law, cryptology, algebra, theory of equations, among first to employ letters for unknowns.
1550-1617	John Napier—*Logarithms*, computing rods (Napier's bones), algebra, spherical trigonometry.
1558	Elizabeth made Queen of England, died 1603.
1560-1630	Henry Briggs—*Logarithms*, navigation, trigonometry.
1603	James I made King of Great Britain.
1607	Settling of Jamestown.
1608-1674	John Milton.
1608	Telescope was invented.
1574-1660	William Oughtred—algebra, *slide rule*, logarithms.
1624	Dutch West India Company commenced settlement of New York City.
1650	New York City was called Nieu Amsterdam and had a population of 10,000.
1581-1626	Edmund Gunter—astronomy, first table of log sines and log tangents to common base, invented surveyor's table and chain, kind of slide rule known as Gunter's scale.
1598-1647	Bonaventura Cavalieri—pupil of Galileo, conics, trigonometry, optics, astronomy, recognized value of logs, principle of indivisibles.
1596-1650	Rene Descartes—analytic geometry (graphic treatment of eq.)
1638	Harvard founded.
1654	Louis XIV crowned.
1608-1665	Pierre de Fermat—theory of numbers.
1623-1662	Blaise Pascal—Pascal's triangle, probability, physics.

[1] $9.15\times10^{7}=91500000$.

Pure geometry is not dependent upon any system of number notation so we find its roots in the distant past; but trigonometry, certain phases of astronomy, algebra, and physics are. We realize upon glancing at the foregoing list of notables and events that a greater and greater amount of sheer calculation was becoming necessary to work with the new principles set forth in the 16th century. When Napier's treatise on logarithms first appeared, multiplication and division of numbers were considered difficult operations and constituted a part of the university course. Just before his time there was a great damming up of "nuisance work", the pressure of which was certain to break through somewhere, John Napier opened the spillway a little in 1614 with the publication of his *Descriptio* and Henry Briggs relieved the pressure further in 1616 by printing logarithms to the base 10. Let us put the case in Napier's own words:

"Seeing there is nothing (right well beloved students in mathematics) that is so troublesome to mathematical practice, not that doth more molest and hinder calculation than multiplications and divisions, square and cubical extractions of great numbers, which, besides the tedious expense of time, are for the most part subject to many slipping errors ... I found at length some excellent brief rules. But amongst all, none more profitable than this, which, together with the hard and tedious multiplications, divisions, and extractions of rootes, doth also cast away from the worke itself even the very number themselves that are to be multiplied, divided, and resolved into roots, and putteth other numbers in their place, which perform as much as they can do, only by addition and subtraction, division by 2 or division by 3; ...."[1]

John Napier and Henry Briggs became fast friends as a result of their work on logarithms. Napier, the Laird of Merchiston Castle (then *outside* Edinburgh) held no civil position because he preferred to be free to travel and live the life of a scholar. Briggs who was the professor of Geometry at Gresham College, London, wrote "... and as soon as the season fo the year and the vacation of my public duties of instruction permitted I journeyed to Edinburgh, where, being most hospitably received by him (Napier), I lingered for a whole month...."[2] We can imagine that these two men had an exciting time intellectually. From here on the story is one of rapid spreading of the theory and use of logarithms. Kepler, the great astronomer, did much to further their use, and tables were published in China in 1650. Edmund Gunter published the first table of log sines and log tangents in 1620.

[1] From *Mirifici Logarithmorum Canonis Descriptio*, Edinburgh, 1614.

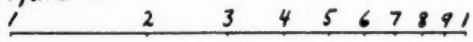
[2] In preface of Briggs' *Arithmetica Logarithmica*.

As we know, if a *series* of numbers forms a geometric progression, the logarithms of these numbers form an arithmetic progression.[1] Thus the addition of logarithms replaces the multiplication of numbers in computation. Now the Greeks studied series but never noticed the relationship between these two series, undoubtedly because of their complicated and clumsy notations.

Just as the abacus made calculations simpler in the days gone by, the slide rule was developed to shortcut the work of logarithms. This instrument has not always been the neat little tool produced now by such companies as Keuffel and Esser or Dietzgen; it had its period of evolution.

In 1620 Edmund Gunter designed his logarithmic scale of numbers on which the distances were proportional to the logarithms of the numbers indicated. Distances were added and subtracted with the aid of compasses, thus multiplication and divisions were performed. (See figure 5).

Figure 5



In 1622 William Oughtred added the inverted scale to be used with the original; therefore, compasses were discarded. In 1850 Amedee Mannheim, a French artillery officer, designed a rule with he Amedee Mannheim, a French artillery officer, designed a rule withe slide movable between two parts of a rigid stock. This is really the grand-daddy of our present rules. Oh, lengths of scale have increased modern methods ct manufacture have produced more accurate rules, and special rules have been designed for specific trades and professions,[2] but fundamentally, they are all very much alike.

We accept modern notation, these slide rules, the Burrough's adding machine and the Monroe calculator without question or appreciation. The modern seventh grader can calculate faster than did Pythagoras and the high school student with his slide rule can out log Napier. What next?

[1] An arithmetic series is formed by the addition of the same number, to form succeeding terms.

A geometric series is formed by the multiplication by the same number, to form succeeding terms.

2+4+6+8+ Also note 2^2, $2j$, $2j$, $2j$, $2j$ { to multiply 4x16, add 2+4 and
2+4+8+16+ 4, 8, 16, 32, 64 { read 64, the result.

[2] One catalog lists Stadia slide, rule, surveyor's duplex, Roylance electrical, chemist's duplex, and others.

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Dr. Ernest E. Blanche, of 9409 Montgomery Avenue, Worth Chevy Chase, Maryland, has been appointed to the teaching staff of the Army University organized by the War Department for American veterans at Florence, Italy. Dr. Blanche is on leave from the Foreign Economic Administration where he is principal statistician. He left for Italy on August 12 to assume his duties teaching mathematics and statistics. He has the assimilated rank of colonel.

Dr. Blanche formerly taught at the University of Illinois, Michigan State College, and the University of Buffalo. He also served two years directing statistical activities at the Curtiss-Wright Corporation, Airplane Division, Buffalo, New York.

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Whither Mathematics?

By HARRY M. KEAL
Cass Technical High School
Detroit, Michigan

Mathematics has proved to be the greatest stumbling block in the path of the so-called Modern Progressive Education. When the teaching of mathematics was subjected to the diluting experiments of our educational theorists, the pitiful results came to the surface immediately because of the cumulative and sequential nature of the subject. What was the reason behind the new theories of education? What caused the dilution of studies until the lack of fundamental skills became forced upon public attention? Back in the days of our pioneers, culture and higher education were limited to the clergy, the jurists, and the well-to-do. The mathematics of algebra and geometry were cultural subjects reserved for those who were fortunate enough to be able to plan for college and the professions. As high schools developed, these cultural subjects became part of the standard curriculum, but they were purely academic and little or no attempt was made to connect higher mathematics with the practical problems of daily life. As secondary education became more universal, the justification of these cultural subjects was questioned. A foreign language, living or dead, was still necessary for college entrance, and the mathematics of geometry and algebra were needed for entrance in engineering and scientific courses. The standards of the college group were of necessity kept; changes were made to fit the non-college group. They became the guinea pigs for the experiments of our educational theorists because their future would never test the value of their education. They were the future hewers-of-wood and drawers-of-water who were to work with their hands and not with their heads. If they were taught to become good citizens with a smattering of the three R's, other education would be wasted upon them. So the dilution theory was invented. The college preparatory group was to be made to work,

and the rest would be encouraged to do their best. There would be few or no failures, and parents and children would all be happy.

The dilution of subject matter thus came about as an unexpected result rather than as a planned objective. In fact, it came from having no objectives for the non-college group. The pupils found the practice of self-expression, interest, motivation and activation a beautiful adventure. By simulating interest in the school projects, they could avoid study. Boys and girls at home as well as in school have always been ingenious in postponing and avoiding work. When the school game came to a pause as a project was completed, everyone had had a good time even if it were difficult to measure and evaluate the educacational gains. I know of one class in geometry that investigated the geometrical applications in newspaper clippings for a whole semester and developed *one* geometric truth. At this speed, assuming geometry to be valuable, a full lifetime would be required to gain a usable knowledge of the subject. The over-emphasis of devices for teaching, measured pleasures instead of measured skills, minimization of the importance of subject matter, films instead of facts, all brought about the dilution of basic subjects to an alarming degree. The three R's suffered with the rest until they became the three I's of Illiteracy, Illegibility, and Inaccuracy. The popularity of the non-college group with its easy ways has increased until it numbers eighty to eighty-five percent of the enrollment of secondary schools. The public, in spite of the efforts of educators in the press and on the air, are beginning to realize that their children are not prepared in the basic skills which have always been considered essential. Class-room teachers have made an honest effort to cooperate with the ever-changing methods, but they are hampered by discipline problems caused by self-expression and no repression. A few can always be found who will seek promotion by calling the training successful, but the average teacher will keep silent when questioned and shake her head in despair.

Consider the pioneer days when the three R's were sufficient for the masses. Industry was in the home. Spinning and weaving were home work. The community blacksmith was the metal worker. Measurements were simple and crude. Transportation was by stagecoach. Communications were confined to stagecoach mail. Who could envision of progress of the next fifty years and the needs of a more complicated education for the common people? The steam engine, the cotton gin, the sewing machine, the spinning jenny, the machine loom, the steam locomotive, the telegraph, farm machinery, and many other inventions brought manufacturing out of the home and into city factories and changed an agricultural nation into an industrial power. Science and further invention flourished. Engi-

neering became a profession. The training of scientists and engineers now demanded mthematics not as a cultural subject but as a positive necessity for the work to be done. The workman in the factory became an artisan whose work required greater and greater accuracy of measurement and his knowledge of arithmetic was extended to include the use of the formulas in handbooks. So algebra became a necessary tool in an industrial age. Our educators have had fifty years to discover this fact, but some of them are still ignorant of it.

The development of machine tools completed the picture of industrial mathematics. Working constantly with lathes, drills and other rotating machine tools made the geometry of the circle a prime necessity, while machining at an angle brought the geometry of the triangle and trigonometrical measurements into the training of the skilled mechanic. All this came at a time when arithmetic was still considered as a serious subject and had not as yet been diluted. Mechanics were drawn from the non-college group and their foundation training was sound. Corporation schools were founded which built upon this foundation and trained their own skilled labor. About the beginning of the century, certain schoolmen saw the advantage of mechanical training and manual training became popular. It was very simple in its scope and consisted of training in the use of carpenter's tools and a little sheet metal work. The instruction had the merit of training the pupils in the use of their hands and instilled in them the necessity of accuracy of measurement. The instructors were often men who had had little college training, and the pupils were usually drawn from those who were failing in subjects requiring much study. As a training for industry, it had little or no value. With the automobile, the demand for technical training rose to new heights. Mass production and interchangeable parts took the place of fitting parts together as they were produced. Refined measurements made necessary the manufacture of gages of every description. The ratio of skilled workmen to unskilled labor became greater day by day. The training of this skilled labor overtaxed the capacity of the corporation schools and and industry turned to public education for help. Technical high schools were the result. The increasing demand for more and more pleasure cars completed the industrialization of the country. People had found the answer to the disposition of their spare time. America was on wheels and she was later to sprout wings. But her children would have to prepare for the changes by being trained in the basic skills and knowledge of physics, chemistry, biology, mathematics and English. These subjects would have to be taught, not as cultural subjects alone, but as a foundation for their applications in the busy life of a people engaged in bringing new comforts and luxuries into their homes.

The demand for mathematics in industry does not rest entirely upon manufacturing but upon change. If an industry could remain static, production could depend upon a minimum of skilled labor which could be recruited from a small predetermined group of fairly high intelligence. This would be the maintenance and repair group whose duty it would be to keep the standardized machinery and equipment in working order, the work being done by unskilled labor. But modern industry is not static. Progress demands change in striving for greater perfection. The people demand greater efficiency, greater refinement and more beauty in the products of industry. Competition makes change more rapid. So skilled labor is demanded in three fields of industry, all of which are constantly active in normal times. The first is the field of new design and "tooling-up"; the second is the field of maintenance and repair in production; the third is the sales and service field. The skilled men in the first group are planning next year's model; in the second group, they are keeping the machinery of this year's model in operation until the new one is ready; and in the third, they are selling the current model and used models of all years while planning a selling campaign for the model being planned. The third group also keeps all models in running order—the service men.

The automotive industry is the best illustration of modern demands for skilled labor, for it led the way to mass production and the grouping of unskilled labor on production machinery and assembly lines. When visitors are conducted through a modern mass production factory, they see only the production group, each man repeating the same small part of the work, and they conclude that no educational training is required. They do not see the skilled groups at work. The public is not shown the new model until it is placed on sale. No change is in sight. They see a ready market for that school product which fails to respond to educational methods. Only the marvel of timing and planning that is mass production is shown to visitors. Any change in a vital part of an automobile demands other corresponding changes. A change in motor speed demands a change in the gears delivering the power to the wheels. A change in the wheel base will call for a complete replanning of the steering mechanism so that the wheels will travel in concentric circles on curves. A change in the octane content of the fuel will call for a change in the size of the combustion chamber. In the field of aviation, a change in the position of the motors will necessitate a complete change of the structure carrying the modified leverage.

The writer has had the good fortune to watch the unfolding of modern mass production of automobiles in the city which is recognized as its center. His work has been to develop the mathematics necessary for the training of the skilled workman in a great industrial com-

munity. Part of that work has been with adult workmen who attend evening classes in order to attain a higher level of skill. In this way, many problems have been brought to the famous technical high school with which he is connected. From these thousands of men, from the various curriculums of the school and from the industries paralleling these curriculums, the mathematical needs of the industrial community have been defined and arranged in sequence. Throughout the years of industrial progress those needs have multiplied until practically all of the traditional steps of the old academic mathematics have become necessities in the equipment of a skilled workman. Arithmetic, algebra, geometry and trigonometry are now in his kit of tools as usable equipment, and the language of conic sections and calculus are part of his vocabulary. He may not recognize their particular place in mathematics, but he knows about involute gears and cardioidal cams, and has to deal with maxima and minima values. Today, a thorough knowledge of all high school mathematics is necessary for those who hope to reach the higher levels of skilled labor. Basic training in physics and chemistry and science in general (not general science) are also necessary for removing obstacles in the way of future advancement. The automotive industry is only one of many industries needing educational foundation for its future skilled workmen. Radio, television, electronics, aviation, refrigeration, air conditioning, agriculture, hydro-electric power, metroleum products, metallurgy,—the list is hardly begun. The formulas involved demand the use of logarithms, exponential equations, complicated graphs, and expert triangulation. If the demand for mathematics training calls for as much additional study in the next decade as it has in the past decade, educational leaders will have to change their ideas and raise their sights. Mathematics has come into its own.

It is recognized by all except some educational experts that job levels correspond to levels of skill in the fundamentals of education. If an employee has mastered the fundamental operations of common and decimal fractions, he can reach a certain employment level. As he adds the algebra of the formula to this skill, new levels are possible. The truths of geometry, the calculations of the dimensions of inscribed figures, and the areas and volumes of solids, become necessities in the middle levels. The use of natural functions in the solution of both right and oblique triangles, in addition to the finding of distances in machine operation, make trigonometry a part of the equipment of a highly skilled man. In a technical high school, the training in the shops and laboratories is only the application of the mathematics and science learned in the classrooms. Through science and mathematics, the applications of those shops and laboratories

take on their true meaning and form the link between the school and the industrial life of the community. Without science and mathematics, shop classes would be only production training for low skill jobs. In this statement lies the definition of technical education. If the shop training is linked to the trade it represents, by a knowledge of the corresponding scientific laws underlying the making and operation of the trade machinery, and by mastery of the mathematics necessary for this knowledge, the student can become a master tradesman. He then has had true technical training.

During the depression, when change was at a minimum, new skilled workmen were not needed in any great numbers and basic education suffered a corresponding slump. Our schools lowered standards in the fundamentals, did away with drill, created a laboratory-less science, an effortless mathematics, a grammar-less English, and founded the high dilution theory of Progressive Education. Just as this easy education had been sold to the public and had permeated the teachers colleges, along came the greatest change-producer of all times, World War II. On top of the immediate demand for skilled labor in making the turnover for the production of war materials, came the revelation that our youth, when called to the colors, were not prepared to take over the operation and care of the machinery of a highly mechanized warfare. They needed the same careful training in mathematics that the technical schools had provided, but which had been neglected in general education. In other words, they needed the mastery of number combinations which their grandfathers had so valued and which modern education had failed to give them. There had been so much gloom from unemployment and its effects that our educators felt that eliminating failures from school life would raise the spirits of their patrons. Compulsory education had crowded the secondary schools with those who normally would have dropped out and with these, many of low mentality. So the educational machinery was geared to the new type of student, and standards were lowered to fit his mentality. Pupils in secondary schools were enrolled in two sharply divided groups, the college preparatory and non-college preparatory divisions. The college preparatory students were drawn from those of higher mentality and were held to a fairly high standard of scholarship. The others were a mixed group composed of those of lower rating, those not planning to go to college, and the ever growing group of those who wished to avoid work and study. The college group was constantly depleted by the transfers to the non-college group of those who were looking for diplomas without work. In many high schools, less than fifteen percent of the pupils are now enrolled in college preparatory courses and the schools have had to

lower standards in these courses to meet the competition. As for the non-college group, they can choose their own subjects and graduate from high school without a single usuable skill in the fundamentals. No high school mathematics is required for graduation. So algebra, geometry, and trigonometry have gone back to their position in the horse-and-buggy days so scorned by our modern educators. They are reserved for the few and denied to the ones who need them the most—the future skilled workman. For, from this class are drawn the industrial workers of the future, that is, from among those who do not plan to enter the professions through college training.

This non-college group contains all grades of mentality. Our educator friends used to tell us that all pupils grouped according to their mentality would fall into a normal distribution with approximately as many of A-mentality as of E-mentality. Now the same educators claim that sixty per cent of all pupils entering high school are incapable of doing work of high-school grade. Some are born stupid, some achieve stupidity, and some have stupidity thrust upon then by inadequate methods of education. All but the very lowest mentalities are capable of learning the number combinations of arithmetic necessary for the lowest levels of skilled labor; but they are not learning them. All of the medium mentalities are capable of learning numerical algebra, geometry, and trigonometry, necessary for the higher levels of skilled labor, but they are not learning them. If the highest type are attracted to engineering,s,cience, and the professions, from whence will come our future skilled technicians? Our future teachers are being trained according to the theory that promotion of pupils is not the reward of study or of reaching certain levels of skill. Our pupils are being taught that they can obtain credit by pretending interest and avoiding work. Our administrators are preaching the importance of methods of teaching and the relative unimportance of the basic skills upon which modern business and technology depend. The product of our secondary schools has been found deficient in elementary skills by the training centers of the armed forces; industrial leaders have long since given up hope of finding mathematical intelligence in the graduates of general education; and the parents of the nation have awakened to the fact that their children were being allowed to graduate from high school without adequate preparation for the business of life. The college preparatory group is fast degenerating into mediocrity with superficial training in those fundamentals which have been thought to be the necessary equipment for the literary, intellectual, and professional leaders of the past.

The writer is not guessing about the mathematical needs of industry. He has had problems brought to his department from many

different industries of this section of the country. These problems mark the different levels of skill of the workmen involved and also show the growing demand for higher levels of mathematical training as the products of industry become more refined and accurate in their measurements. Practically every step in the natural development of the old time classical algebra is now justified by present industrial needs of skilled labor below the grade of engineer. The truths of plane geometry and the ability to comprehend the drawings of a solid in perspective or from different angles are necessary equipment and are best learned through high school geometry. The best texts on shop mathematics now published presuppose a knowledge of algebra, geometry, and practical trigonometryl The evidence is on every side of us. It only remains for us to convince those educators who, themselves untrained in science and mathematics, are blind to the trend of the industrial times.

The picure is not all dark. Certain of our educational leaders have recognized that there is something wrong, and a few have begun to preach the gospel of adequate education. With proper backing from those of us who still believe in the future, from the parents of our youth and from those who have a clear picture of our changing industrial world, we will see mathematics begin to recover from its present lowly position. The demand for mathematics training exceeds the supply and that need is growing while the supply is diminishing. Technical high schools have been unable to supply the demand. Employers report the scarcity and employes are trying to fit themselves for higher levels in evening adult classes. The diminishing supply of mathematically trained high school graduates is due to two perfectly apparent causes: the training of instructors in our teachers colleges and the administration of our school systems. When professors of education, themselves confessedly deficient in mathematics, take pride in their ignorance before the future teachers in their classes, then education in our country is on the wane. When the administration of school systems allows the dilution of basic science and mathematics to its present content, it is time for a protest from those who wish our country to maintain its leadership.

Fermat's Integration of X^n

By Carl B. Boyer
Brooklyn College

Elementary courses in the calculus now customarily introduce the derivative before the integral; but historically the situation was the reverse. The quadratures of Archimedes preceded the differential tangent-methods of Fermat by some nineteen hundred years. For ordinary functions, however, the technique of differentiation is simpler than the direct method of quadratures. Hence when Newton and Leibniz discovered and developed the inverse relationship between the derivative and the integral, differentiation was made the primary operation and integration became the inverse problem of determining an anti-derivative or "integral in the sense of Newton." Nevertheless, ever since the time of Cauchy the concept of the integral as a limit of a characteristic sum has played an increasingly important role in the development of analysis. For this reason it is desirable, even in a first course in the calculus, to emphasize this idea of the "integral in the sense of Leibniz." Recently a number of writers have adopted this point of view to the extent of presenting the integral-sum before the antiderivative, or even before the derivative itself. One disadvantage in attempting such a program has been the difficulty of carrying out this process of integration even for ordinary functions. In this connection authors of textbooks seem to have overlooked an admirably simple and appropriate example given by Fermat just about three hundred years ago.*

The class of functions $y=x^n$, where n is a positive integer, was historically the first to be differentiated and integrated. It is still the first type of function to be differentiated by beginners in the calculus; and it is pedagogically natural and desirable to apply the direct process of integration to such expressions. The method of Fermat is here probably the simplest approach, independent of differentiation, in evaluating

$$\int_0^a x^n dx.$$

*The date of Fermat's discovery of this method is uncertain. He seems to have been in possession of the *result*—at least for positive integral powers—by 1636; but this appears to have been based on considerations from the theory of numbers involving the sums of powers of the integers. Inasmuch as he had found the quadratures of

To illustrate his procedure,* construct the curve $y=x^n$ and mark off on the x-axis the points with abscissas a, ae, $ae^2, \cdots$, where e is any positive constant less than one. (See Fig. 1) At these points erect

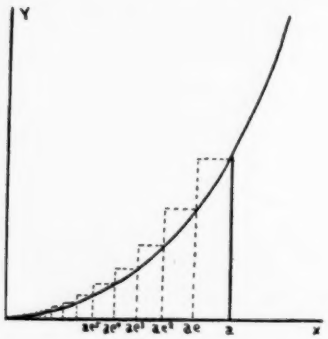

Fig. 1

ordinates to the curve and, beginning with $x=a$, construct rectangles on successive ordinates and absciss-segments. The areas of these rectangles form a geometric progression

$$a^{n+1}(1-e)[1+e^{n+1}+e^{2(n+1)}+e^{3(n+1)}+\cdots].$$

The "sum to infinity" of this series is $a^{n+1}(1-e)/(1-e^{n+1})$ or $a^{n+1}/(1+e+e^2\cdots+e^n)$. As e tends toward unity—that is, as the area of each rectangle tends toward zero—this sum approaches as a limit $a^{n+1}/(n+1)$, the value of the integral desired.†

Fermat's method of quadratures is well adapted to classroom use, for it is clear and simple, yet adequately rigorous at that level. Historically it was, in fact, logically the most satisfactory evaluation of

$$\int_0^a x^n dx$$

given before the nineteenth century.

"parabolas of fractional degree" by 1644, it is probable that by this date he was in possession of the general method presented in this paper. However, the definitive redaction of his work in this connection apparently was not made until 1657.

*The exposition of Fermat differs radically from that given in this paper. Fermat did not adopt the Cartesian notation for equations, but used instead the older syncopated algebraic forms of expression. Moreover, the analytic viewpoint and the function concept had not yet displaced the synthetic form of ratios and proportions, and this fact adds to the difficulty of reading works of the time.

†See Pierre Fermat, *Oeuvres* (ed. by Pail Tannery and Charles Henry, 4 vols. and supplement, Paris, 1891-1922), I, 255-288; III, 216-240. Fermat did not here use the words limit and integral, but his general notions are equivalent to these.

The method above can be easily generalized, as Fermat himself showed, for other values of n. For negative integral values, $n \neq -1$, one evaluates

$$\int_a^\infty x^{-m}dx$$

by choosing $e>1$ and setting up the approximating rectangles as in the diagram (Fig. 2). The corresponding geometric progression is

$$a^{1-m}(e-1)[1+e^{1-m}+e^{2(1-m)}+e^{3(1-m)}+\cdots],$$

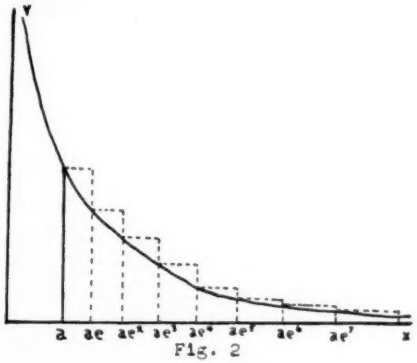

Fig. 2

of which the "sum of infinity" is $a^{1-m}(e-1)/(1-e^{1-m})$ or

$$-a^{1-m}e^{m-1}/(1+e+e^2+\cdots+e^{m-2}).$$

As e tends toward unity, this sum approaches $a^{1-m}/(1-m)$.

For rational fractional values

$$n=\frac{p}{q}$$

(except values between -1 and 0), the procedure is substantially that outlined above after one has made the substitution $e=K^q$. One then writes

$$\frac{1-e}{1-e^{p+q/q}}$$

as

$$\frac{1-K^q}{1-K^{p+q}}=\frac{1+K+K^2+\cdots+K^{q-1}}{1+K+K^2+\cdots+K^{p+q-1}},$$

and this ratio approaches

$$\frac{q}{p+q}$$

as e (and hence also K) tends toward unity. By applying the relationship

$$\int_r^t f(x)dx = \int_r^s f(x)dx + \int_s^t f(x)dx,$$

where $0<r<s<t$, one can easily find the definite integral of x^n between any positive (or negative) limits and for all rational values of n for which $-1>n>0$.

For $n=-m=-1$, the areas of the approximating rectangles (for a given value of e) are equal. The improper integral

$$\int_a^\infty x^n dx$$

consequently has no value; but it was known in Fermat's day* that in this case the quadrature between finite positive limits is given by logarithms, inasmuch as the area under the curve increases in arithmetic proportion while the abscissas increase in geometric proportion. For values of n between 0 and -1, the area under the curve $y=x^n$ is again infinite, but in these cases the value of

$$\int_a^\infty x\,dy$$

can be determined as above by the method of Fermat.

*For an altogether excellent account of Fermat's quadratures and their place in the work of that time, see H. G. Zeuthen, "Notes sur l'histoire des mathématiques," *Oversigt over det Kongelige Danske Videnskabernes Selskabs. Forhandlinger*, 1895, pp. 37-80; or see his *Geschichte der Mathematik im XVI. and XVII Jahrhundert* (transl. by Raphael Meyer, Leipzig, 1903), especially pp. 248-300.

Brief Notes and Comments

Edited by
MARION E. STARK

17. *Multiple integration.* In teaching multiple integration, it is often desirable to show an example of a double integral which can only be evaluated in terms of elementary functions by one choice of the iterated integral. The purpose of this note is to exhibit several of these.

I. Integrate $e^{y/x}$ over the triangle with vertices (0,0), (1,0) and (1,1). Ans. $(e-1)/2$.

The iterated integrals are

$$\int_0^1 dx \int_0^x e^{y/x}\,dx \quad \text{and} \quad \int_0^1 dy \int_y^1 e^{y/x}\,dx.$$

The second of these cannot be evaluated in terms of elementary functions.

II. Integrate $(y \log x)^{-1}$ over the region in the first quadrant bounded by the parabola $y=x^2$, $x=y^2$. Ans. $-3/2$.

III. The integrands of the previous examples have had isolated singularities on the boundaries of the regions. This is not the case with this one, perhaps better phrased as a problem in moments with variable density.

Integrate ye^{xy} over the triangular region bounded by the hyperbola $xy=2$, and the lines $x=2$, $y=2$. Ans. $(e^4/2)-(3/2)e^2$.

Integrating first with respect to x, and then y, the evaluation is simple. The other order of integration can also be done, but requires a trick that most students might very well not think of.

Many other examples may be constructed along the same lines.

Society of Fellows, Harvard University. R. C. BUCK.

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18. The following letter, written by B. C. Patterson of Hamilton College, Clinton, New York to N. A. Court of the University of Oklahoma, Norman, Oklahoma, is printed here, with the permission of both the above named gentlemen, because it was felt that the contents

might be of interest to readers of the MAGAZINE. Professor Court's article on "Elements at Infinity in Projective Geometry" appeared in the December, 1944, issue of the NATIONAL MATHEMATICS MAGAZINE.—ED.

DEAR DR. COURT:

I wish to thank you for your kindness in sending me reprints of your recent mathematical articles. I have read them, particularly "Geometry and Experience" and "Elements at Infinity in Projective Geometry", with interest and enjoyed them.

In the last one mentioned your statement that the difference between the Euclidean and projective line (and plane) is purely verbal called to mind discussions I used to have with an elderly friend, a retired professor of physics, who attended my projective geometry class several years back. It seems to me that those entities, the Euclidean point and the projective point, originally undefined, are later, to some extent, defined as to their properties by the assumptions they have to fit. One might maintain that the difference between the two sets of assumptions are purely verbal but I should feel it is stretching the idea of "verbal" somewhat to say that the entities themselves have only verbal differences. Euclidean points, for example, may have a "between" property if there are three of them; but what of projective points? Similarly, the invariant called "distance" belongs to the former kind but not the latter, etc.

I have certainly condensed a highly philosophical discussion in the above paragraph, and that is more than often not a futile thing to do! Let me say again, however, that I enjoyed reading the reprints and thank you very much for sending them.

Very sincerely yours,

BOYD C. PATTERSON.

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19. The following factorization of $2^{135}-1$ has been received from Professor Emory P. Starke, to whom it was sent by M. Victor Thébault (Tennie, Sarthe, France), who in turn received it from Monsieur P. Poulet (Saint-Omer, France):

$$2^{135}-1 = 7 \cdot 31 \cdot 73 \cdot 151 \cdot 271 \cdot 631 \cdot 23311 \cdot 262657 \cdot 348031 \cdot 49971617830801.$$

M. Thébault adds (on his card dated March 15, 1945) that, in spite of suffering, French scholars are continuing courageously with their research.—ED.

The editor acknowledges with gratitude the assistance of Professor Lennie P. Copeland and Dr. Katharine E. Hazard who have acted as referees for this department of the MAGAZINE during 1945.

Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to E. P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 579.* Proposed by *Howard Eves*, Syracuse University.

Find an expression for the length of an altitude of a tetrahedron in terms of the three edges and the three angles at the vertex through which the altitude is taken.

I. Solution by *P. D. Thomas*, U. S. Navy.

Let the tetrahedron be $V-ABC$, where VA, VB, VC are respectively a, b, c, and angles BVC, CVA, AVB are respectively α, β, γ, where h is the altitude from V, and where

$$K=[s(s-AB)(s-BC)(s-CA)]^{\frac{1}{2}}, \tag{1}$$

with $s=(AB+BC+CA)/2$, is the area of ABC. The volume of $V-ABC$ is

$$hK/3=abc(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)^{\frac{1}{2}}/6, \tag{2}$$

(See R. J. T. Bell, *Coordinate Geometry of Three Dimensions*, p. 28.) By the law of cosines

$$AB=(a^2+b^2-2ab\cos\gamma)^{\frac{1}{2}}, \quad BC=(b^2+c^2-2bc\cos\alpha)^{\frac{1}{2}}, \quad CA=(a^2+c^2-2ac\cos\beta)^{\frac{1}{2}}. \tag{3}$$

The required expression is obtained by substituting these values from (3) into (1) and the result into (2). It is also possible to eliminate the three cosines and obtain h expressed in terms of the lengths of the six edges.

*From the issue of November, 1944. Another No. 579 appeared in the October, issue: its solution may be found in the issue for May, 1945, p. 423.

II. Solution by the *Proposer*.

Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be unit vectors along VA, VB, VC respectively, and $\bar{n}$ the unit vector along h. Let n_a, n_b, n_c be the magnitudes of the components of $\bar{n}$ along VA, VB, VC respectively. Also, set $a' = 1/a$, etc. Then we have

$$\bar{n} = n_a\bar{a} + n_b\bar{b} + n_c\bar{c}, \tag{1}$$

$$(\bar{n}\cdot\bar{a})a = (\bar{n}\cdot\bar{b})b = (\bar{n}\cdot\bar{c})c = h, \tag{2}$$

$$\bar{b}\cdot\bar{c} = \cos\alpha, \quad \bar{c}\cdot\bar{a} = \cos\beta, \quad \bar{a}\cdot\bar{b} = \cos\gamma. \tag{3}$$

From (1) and (2) we have

$$[(n_a\bar{a} + n_b\bar{b} + n_c\bar{c})\cdot\bar{a}]a = h,$$

or

$$n_a + n_b\cos\gamma + n_c\cos\beta = ha'.$$

Similarly we have

$$n_a\cos\gamma + n_b + n_c\cos\alpha = hb',$$

$$n_a\cos\beta + n_b\cos\alpha + n_c = hc'.$$

Solving for n_a, n_b, n_c by Cramer's rule, we get

$$n_a = h \begin{vmatrix} a' & \cos\gamma & \cos\beta \\ b' & 1 & \cos\alpha \\ c' & \cos\alpha & 1 \end{vmatrix} \div \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}, \text{ etc.}$$

But

$$\begin{aligned} 1 &= \bar{n}\cdot\bar{n} \\ &= \bar{n}\cdot(n_a\bar{a} + n_b\bar{b} + n_c\bar{c}) \\ &= n_a(\bar{n}\cdot\bar{a}) + n_b(\bar{n}\cdot\bar{b}) + n_c(\bar{n}\cdot\bar{c}) \\ &= n_a ha' + n_b hb' + n_c hc'. \end{aligned}$$

After substituting the values of n_a, n_b, n_c and simplifying the resulting coefficient of h^2, we obtain finally

$$\begin{vmatrix} a' & b' & c' & 0 \\ 1 & \cos\gamma & \cos\beta & a' \\ \cos\gamma & 1 & \cos\alpha & b' \\ \cos\beta & \cos\alpha & 1 & c' \end{vmatrix} h^2 = \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}.$$

Notes. (1) The result established above gives the perpendicular distance from the origin to a plane whose intercepts are a, b, c and where the coordinate axes are oblique. (2) If $\alpha = \beta = \gamma = 90°$, we have $h^2 = 1/(a'^2 + b'^2 + c'^2)$. (See N. A. Court, *Modern Pure Solid Geometry*, p. 92, Art. 285.) (3) The analogous formula for the plane also holds. (4) The problem arose in the calculation of interplanar spacings of crystal systems.

No. 581. Proposed by *N. A. Court*, University of Oklahoma.

Given a coaxal pencil of spheres, the circle of intersection of a variable sphere of the pencil with the plane passing through a fixed line of the radical plane of the pencil and through the pole of the radical plane for the sphere considered, lies on a fixed sphere.

Solution by *J. S. Guérin*, Catholic University of America.

Let (c) be the basic circle, real or imaginary, and (R) the radical plane of the given coaxal pencil, m the fixed line in (R), and P the pole of (R) for the variable sphere (S) of the pencil. If (M) is the sphere orthogonal to (S) and passing through the circle along which (S) is cut by the plane $P-m$, the center M of (M) is the pole of the plane $P-m$ with respect to (S), hence M lies in the plane (R), and therefore M is the pole of the line m for the circle (c).* Thus as (S) varies, the point M remains fixed, and so does the sphere (M),† hence the proposition.

Also solved by *Howard Eves* and *Paul D. Thomas*.

EDITORIAL NOTES. 1. Thomas assumed that the coaxal pencil is non-intersecting. He denotes by a and b the distances from the center of (c) to a limiting point and to the fixed line m, respectively, and finds that the square of the radius of the sphere (M) is equal to $a^2(a^2+b^2)/b^2$.

2. The center O of the circle of intersection of the variable sphere (S) with the plane $P-m$ is the inverse of M with respect to (S), hence the locus of O is a circle, as has been pointed out by both Guérin and Thomas (see Court's *Modern Pure Solid Geometry*, p. 191, ex. 18).

3. The fixed sphere (M) is always real, if the given coaxal pencil is non-intersecting. In the case of an intersecting pencil, (M) is real or imaginary according as the given fixed line m does or does not cut the basic circle (c) in real points. If m is tangent to (c) the sphere (M) reduces to the point of contact.

4. Eves points out that the analogous theorem in the plane is also true. The proposition in the plane has, in fact, been considered in the *Journal de Mathématiques élémentaires*, 1883, p. 179, Q. 89. —N. A. C.

*Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 141, Art. 426. The Macmillan Co., 1935.

†*Ibid.*, p. 179, Art. 567.

No. 583. Proposed by *Paul D. Thomas*, U. S. Navy.

Construct a triangle given the lengths of the median, the internal bisector, and the ex-symmedian issued from the same vertex. (Note: an ex-symmedian is the segment of a tangent to the circumcircle of a triangle at a vertex between the vertex and the respectively opposite side).

Solution by *R. M. Swesnik*, student, University of Oklahoma.

Let AA', AU, AL be the lengths of the median, the internal bisector, and the ex-symmedian issued from the vertex A of the required triangle ABC.

The circle (L) with an arbitrary point L as center and the given length AL as radius is the Apollonian circle of ABC relative to the vertex A, hence (L) passes through the foot U of the bisector AU. Thus if in (L) we lay off a chord of length AU. the point A is a vertex of the required triangle ABC, the other two vertices B, C of which lie on the line LU.

Let the circle with A as center and the given length AA' as radius meet the line LU in A'. The perpendiculars at A, A' to the lines AL. LU, respectively, meet in the circumcenter O of the required triangle ABC, and the circle with O as center and OA as radius meets the line LU in the required vertices B, C of ABC.

The chord AU of the circle (L) must be smaller than the diameter of (L), i. e., $2AL > AU$.

The line LU meets (L) again in the foot U' of the external bisector AU' of the angle A, and the segments UU' and BC are harmonic, hence the mid-point A' of BC must lie outside of the segment UU' and on the same side of L as the point U. Moreover, the foot of the perpendicular from A upon UU' necessarily lies on the segment UU', hence $AA' > AU$.

If the two indicated inequalities are satisfied the problem has one and only one solution.

Also solved by *Howard Eves, H. E. Fettis, J. S. Guérin, D. L. MacKay*, and the *Proposer*.

No. 585. Proposed by *John H. White*, Summit, New Jersey.

Two islands, A, B, (considered as points) in a circular lake (O) are equidistant from the center O. Determine the point P on the shore such that the trip $AP+PB$ shall be a minimum. Obtain a geometric construction for the point P.

Solution by *C. E. Springer*, University of Oklahoma.

Let the X-axis be taken through O, and parallel to the line AB, the Y-axis through O and perpendicular to AB. Let the acute angle XOA be denoted by α and the acute angle XOP by Θ. Then

$$(1) \qquad z = AP + PB$$
$$= \sqrt{r^2+R^2-2rR\cos(\Theta-\alpha)} + \sqrt{r^2+R^2+2rR\cos(\Theta+\alpha)},$$

where $r = OA$ and $R = OP$. The derivative of z with respect to Θ vanishes for values of Θ which satisfy

$$(2) \qquad \sin^2(\Theta+\alpha)[r^2+R^2-2rR\cos(\Theta-\alpha)]$$
$$= \sin^2(\Theta-\alpha)[r^2+R^2+2rR\cos(\Theta+\alpha)].$$

By trigonometric reduction, equation (2) becomes

$$(3) \qquad \cos\Theta(r\sin\Theta - R\sin\alpha)(R\sin\Theta - r\sin\alpha) = 0.$$

The roots of (3) are given by

$$(4) \qquad \Theta = \pi/2, \quad \Theta = 3\pi/2, \quad \Theta = \arcsin(R\sin\alpha/r).$$

The value $\Theta = \arcsin(r\sin\alpha/R)$ is extraneous, having been introduced by squaring to obtain (2). It is found that the second derivative of z in (1) with respect to Θ is positive for $\Theta = \pi/2$ if, and only if, $\sin\alpha > r/R$. In this case $\Theta = \pi/2$ gives a minimum value for z, $\Theta = 3\pi/2$ furnishes a maximum value, while the third value in (4) does not correspond to a real angle.

In case $\sin\alpha < r/R$, the angle given by the third of (4) yields a minimum value for z. $\Theta = \pi/2$, $3\pi/2$ both correspond to relative maxima. To construct the point P for this case, extend OA to point Q on the circle of radius R and draw a line through Q parallel to AB This parallel intersects the circle of radius r in the point M. The line OM intersects the circle of radius R in the required point P.

EDITORIAL NOTE. The suggestion of *J. H. Butchart* may be amplified into the following argument. Let the ray which bisects angle AOB meet circle (O) in D and consider the circle (C) determined by the three points A, O, B. If (C) and (O) are tangent, the point of tangency is D. If they intersect, let the intersections be Q_1 and Q_2. The required point P will be shown to coincide with Q_1 or Q_2 (which are symmetrically equivalent) when they exist: otherwise it is D.

Consider the ellipse E having A, B as foci and of such size as to lie within (O) and to be tangent to (O)—at S_1 and S_2 (or at D). For every point T on E, $AT+TB$ is constant. Then obviously P coincides with S_1 or S_2 (or with D)since all other points of (O) are outside E. Now the normal to E at any point bisects the angle between the focal radii of that point: since E is tangent to (O), this normal passes

through O. Hence P is such that AO and OB subtend equal angles at P. The locus of points having this property consists of the circle (C) and the line OD. (For points on the external segment AB, the subtended angles are equal but OP is not the bisector). Thus S_i coincide with Q_j, and the required point P on (O) is as stated at the outset. It is easy to show that when points Q_j exist, $P=D$ corresponds to a relative maximum of $AP+PB$.

If the restriction $OA=OB$ is removed, the problem becomes more complicated because the locus of points from which AO and OB subtend equal angles is then part of a strophoid through A and B with its double point at C. (See article by N. A. Court in the *American Mathematical Monthly*, 1915, pp. 20-22; also note to problem E479, 1942, p. 200). The point P on (O) such that $AP+PB$ is minimum is found among the intersections of (O) with this strophoid.

Also solved by *Howard Eves*.

No. 593. Proposed by *Howard Grossman*, New York City.

An array of a vertical lines at unit intervals all intersecting $b(\leqq a)$ horizontal lines at unit intervals contains $b(b-1)(3a^2-9a+2b+2)/12$ non-square rectangles.

Solution by *Emil D. Schell*, Arlington, Virginia.

There are ab possible choices for one of the vertices of a rectangle. Deletion of the horizontal and vertical line containing the selected vertex leaves $(a-1)(b-1)$ choices for the diagonally opposite vertex. These two vertices determine the rectangle, but yield only

$$ab(a-1)(b-1)/4$$

different rectangles, since for each rectangle any one of the four vertices may be selected first.

A square is obtained by this choice whenever the two vertices lie on a 45° diagonal of the given lattice. Consider only the diagonals of slope $+1$. If there are k points of intersection on such a diagonal it can serve as the diagonal of 1 square of side $k-1$, 2 squares of side $k-2, \cdots, k-1$ squares of side 1: in all

$$1+2+\cdots+(k-1)={}_kC_2$$

squares. Now in the lattice there are 2 diagonals carrying 2 intersections each, 2 carrying 3 intersections each, ... 2 carrying $b-1$ intersections each; also there are $a+1-b$ diagonals carrying b intersections each. Hence the number of squares is

$$2({}_2C_2+{}_3C_2+\cdots+{}_{b-1}C_2)+(a+1-b){}_bC_2$$

$$=b(b-1)(b-2)/3+(a+1-b)b(b-1)/2=b(b-1)(3a-b-1)/6.$$

The difference between this and the total number of rectangles gives the number of non-square rectangles as stated in the proposal.

PROPOSALS

No. 619. Proposed by *V. Thébault*, Tennie, Sarthe, France.

The points Q, P are the projections upon the sides of a triangle ABC of the feet B', C' of the altitudes BB', CC'. Show that (a) PQ is parallel to BC, and the distance between the two parallel lines is equal to $S : R$, where S is the area and R is the circumradius of ABC; and (b) the lines $B'Q$, $C'P$ are perpendicular to each other and are equally inclined to the bisectors of the angle A.

No. 620. Proposed by *Edmund Churchill*, Rutgers University.

Establish the convergence or divergence of the harmonic series modified by (a) the omission of all terms whose denominators do not begin with the digit 9; (b) the omission of all terms in whose denominators the digit 9 appears.

No. 621. Proposed by *R. B. Deal*, U. S. Navy, and *A. F. Bernhard*, University of Oklahoma.

A man walks east with constant speed u. His dog, keeping at a constant distance r from the man, runs with constant speed v. If the dog is initially north of the man and running west, find the locus of the dog's path. Also show that the next time the dog is again north of the man, it will have traveled a distance equal to the perimeter of an ellipse with eccentricity $k = u/v = \sin A$ and with semi-minor axis $b = r \sec A$.

No. 622. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Form a square of eight digits which is transformed into a second square upon increasing by unity the second digit from the left.

No. 623. Proposed by *E. P. Starke*, Rutgers University.

Given four concyclic points A, B, O, P such that $AO = BO$. Show that the ellipse through P having foci at A and B, and the circle through P having center at O are orthogonal or tangent according as O, P are on the same side of AB or on opposite sides.

No. 624. Proposed by *Howard Eves*, College of Puget Sound.

The centroid of any arc of a transitional spiral coincides with the external center of similitude of the osculating circles of the extremities of the arc. (Note. A transitional spiral is a curve whose curvature varies directly with the arc length).

No. 625. Proposed by *N. A. Court*, University of Oklahoma.

Four parallel cevians AA', BB', CC', DD' of a tetrahedron $(T) = ABCD$ are divided by the points P, Q, R, S, internally in the same ratio, k. For what value of k will the four points P, Q, R, S be coplanar?

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Bibliography and Reviews

Edited by
H. A. SIMMONS and P. K. SMITH

An Outline of the Calculus. By C. O. Oakley, Barnes and Nobles, Inc., New York. vii+221 pages. $1.25.

This outline is intended to serve either as a textbook or as a supplementary textbook. Only with a great deal of teaching could it be used as a textbook. The text seems to be better adapted to serve as a supplementary text, or as a text for review.

Facility and rapidity of use are marks of the book. In the front of the text the chapter headings are given in tabluar form along with the sections from seventeen calculus texts where these subjects are treated. This device makes for broad and easy reference.

Chapter I consists of the common formulas used in the calculus from algebra, trigonometry, and plane and solid analytics. Also at the end of this chapter thirty-two reference graphs are included. Chapter II is, wisely, a brief one on functions and limits. Chapter III is on "The Derivative." The author gives in this chapter merely the definitions of the derivative and an outline of the Δ-process. No numerical illustrations of the Δ-process are included. Two or three such illustrations would be good. All basic differentiation formulas are derived compactly in chapter IV within the limts of twelve pages. "Applications of Differentiation" are treated in Chapter V. Careful and concise statements on testing for critcal values stand out in this chapter. Chapters VI, VII, VIII deal respectively with "The Differential," "Curvature" and "Indeterminate Forms." Formal integration is covered in Chapter IX within the limts of twenty-one pages. One-fourth of this space is devoted to rational functions. The definite integral and improper integrals are the subjects of Chapter X. The usual applications of the single integral are covered in Chapter XI. "Approximate integration" is a well-placed topic, appearing in Chapter XII. In this chapter the author does not derive Simpson's formula. Chapters XIII through XVIII give the multiple integrals and their applications, partial derivatives, and their applications to three-space geometry, infinite series, expansion of functions, and hyperbolic functions. A set of "Sample Examinations" is given in Appendix A and in Appendix B a table of one hundred integrals is given.

A common mistake is made in Fig. 43 in plotting $\sqrt{x}+\sqrt{y}=\sqrt{a}$. This graph has no infinite branches. It only extends from $(0,a)$ to $(a,0)$. At the top of page 31 the statement is made: "Note that since $\Delta y \to 0$, as $\Delta x \to 0$, the

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \text{ is of the form } \frac{0}{0}.\text{"}$$

The reviewer is of the opinion that no suggestion should be made to a beginning student to set the increments equal to zero; he is interested only in finding the

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Again the "$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is of the form $\frac{0}{0}$"

is not critically stated. In his table of integrals, the author states the conditions on u in the two forms of the integral. However, where he derives one of these forms (leaving the other to the student) the author fails to give any derivations of the inequalities imposed on u. In the use of Simpson's rule, the author might have stated that the integral

$$\int_a^b f(x)dx$$

is given exactly provided $f(x)=a_0x_n+a_1x_n^{-1}+\cdots\cdots+a^n$, $n\leqq 3$. This statement would not have been out of place in view of the rigor of statement of theorems in the text.

This text is written in concise and unusually forceful language by a vigorous thinker. The figures are clear and the format pleasing. The text has a definite place and is recommended for examination looking toward adoption.

Louisiana Polytechnic Institute. P. K. SMITH.

Modern;Operational Mathematics in Engineering. By Ruel V. Churchill. McGraw-Hill Book Company, New York and London, 1944. x+306 pages.

Contour integral representation of functions is well known to be a powerful technique for attacking physical problems. Long in use by physicists, this method is finding ever increasing favor with engineers. Professor Churchill's new book is a valuable exposition of one important branch of this subject, namely, integral representation with the Laplace kernel. Addressed primarily to engineers and physicists, the analysis is as elementary as possible, and the method is illustrated by many applications to important physical problems.

The subject is introduced via the *real* Laplace transformation, and the first four chapters are devoted to its properties and applications. Chapters I and II include the following items: definition of the real Laplace transforms; linear superposition; transforms of power, exponential, trigonometric and step functions; transforms of derivatives and integrals; of periodic functions and their half and full wave rectifications; displacement of function and of transform; and retarded integral superposition. Examples of the solution of differential, difference and integral equations are given. The inverse transform is defined implicitly, and an excellent table of transforms is appended.

In Chapters III and IV, the real transformation is applied to the solution of initial and boundary value problems arising from physical problems. Since only the real Laplace transform has so far been discussed, inversion is still accomplished by using general properties and the tables. The transform is expanded and inverted term-by-term in some cases.

Chapter V is an exposition of those parts of the theory of functions of a complex variable which are needed for developing the complex Laplace transformation. Chapter VI develops the Mellin inversion integral and thus completes the formal presentation of the general Laplace transformation. Chapters VII and VIII again deal with applications, the first with heat conduction and the second with mechanical vibration; these chapters differ from Chapters III and IV in that the contour is now in the complex plane and the inversion integral is used. An example of contour deformation is given, but there is no discussion of the powerful method of steepest descent, which is often useful for approximating a function after its integral representation has been found.

Chapter IX discusses Sturm-Liouville systems. It shows how one such system arises from a physical problem, discusses the Green's function and the characteristic functions and values. However, it gives no physical interpretation of these mathematical objects. Chapter X, the last, treats Fourier sine and cosine transforms.

This is an excellent work and will undoubtedly go into many editions because it lucidly displays mathematical methods of great power in treating important physical problems. The remarks which follow should be regarded as suggestions for making a good book more useful to engineers and physicists, rather than as criticisms of the present edition; these reamarks are based on two widely accepted principles. First, it is advantageous to establish a smooth transition between the known and the new, if possible displaying the new as an extension of simpler concepts with which the reader may reasonably be expected to be more familiar. Second, it is of the utmost importance to interpret mathematical formalism in physical terms, not only for a clear understanding of the problem in hand but also because this often illuminates related but more difficult problems.

With regard to the first mentioned principle, it would seem to be valuable to introduce the linear integral transformation as the limit of a linear transformation of the components of a vector, as the number of components becomes large. As is well known, this allows an advantageous extension of the terminology of geometry which suggestively motivates analysis by giving meaning, when applied to functions, of such terms as parallel, orthogonal, dot-product, component, etc.; for example, the transform of a function becomes the component of that function parallel to a given function, except perhaps for a normalizing factor; again, the Schwarz inequality merely asserts that the dot product of two functions is not greater than the product of their "lengths". So long as this terminology is used only to suggest manipulations there is no question of rigor involved, and it is believed that their introduction would have improved this already excellent book.

The book begins with a special case, the *real* Laplace transformation. The choice of a special case is wise, and perhaps there is a very good reason for this particular choice, here overlooked. However it is believed that the Fourier integral, in exponential form, would have been a more advantageous choice; and it seems very unfortunate that this important special case is not mentioned until past the middle of the book (p. 163) and then only part of a page devoted to it. In the first place nearly every engineer and physicist has some familiarity with the continuous Fourier spectrum of a function, this being a very common and extremely useful tool in electricity, elasticity, sound, light, etc. Second, use of the exponential Fourier integral would make it possible to introduce immediately the inversion integral in explicit form, giving the proof later; this would emphasize the reversibility of the transformation and allow a function and its spectrum to be treated on an equal footing as two different realizatilons of one function, that is, the components of a function in two different frames. However, the real transform has a kernel analogous to a general, rather than unitary, matrix; its reciprocal is not known, and hence use of this transform delays the introduction of the explicit inversion formula and gives to the whole treatment an unfortunate unilateral trend which obscures beautiful symmetry of the subject when approached via a unitary kernel.

Finally, introducing the subject via the Fourier integral would make it possible to establish, early in the discussion, an intimate relation between formalism and physics. For example, Section 7 is entitled "A Theorem on Substitution" and deals with the correspondence between a displacement of the transform of a function and multiplication of that function by an exponential. This abstract theorem has certain formal applications, but understanding of it would be enhanced and its value increased if the Fourier kernel had been used. First, on account of the symmetry between the representation in coordinate and frequency space, the translation of both function and spectrum could be presented in one treatment; second, interpretation of these in terms of physical phenomena is direct and well-known—the first is *retardation*, a phenomena of the utmost importance in wave propagation and in time-delay networks, and the

second is *heterodyning*, widely employed in every radio receiver and countless other electronic devices. But in the case of the real transform, the theorem remains one-sided, no physical interpretation is known, and use of the theorem is therefore restricted to a purely formal task, namely that of reducing the labor of solving certain differential equations.

Another example may be found in studying the spectra of special functions, particularly a finite wave train, with square and exponential envelopes, or merely the envelopes themselves. Out of these studies one comes to a more intimate understanding of the character of a spectrum and reaches the simple conclusion that to build fine detail requires high frequency components, and one learns certain semi-quantitative principles concerning the effect of falsifying a spectrum in order to obtain a simpler integral. He also comes to a clear understanding of the broadening of the spectrum of a wave train or pulse as its length or duration is reduced. If interested in quantum mechanics, he finds a direct connection between this formalism and the Heisenberg uncertainty principle.

It may be argued that the class of functions which are representable is reduced by starting with the Fourier integral. However, this apparent disadvantage can actually be turned to an advantage by discussing convergence factors and cut-offs. For example, an engineer will not object to replacing a unit step function $S(t)$ by $S(t)\exp(-at)$ since these will differ negligibly for a time of order $1/a$ which can be made arbitrarily large by choosing a sufficiently small; or, a function can be chopped beyond some arbitrarily long time by multiplying it by the difference of two unit step functions.

The inversion integral need not be rigorously established in order to be used. Pending the derivation of the Mellin inversion in Chapter VI, the inversion integral might very well be rendered plausible by introducing the delta function, which is analogous to the unit matrix in discrete transformations and enters consistently as the dot-product of any two characteristic vectors in unity function space. While there may be some question as to the rigor of the concept of the delta function, there can be none as to its usefulness; and, in many event, the rigorously derived inversion relations behave as if the delta function were a rigorous mathematical concept. It might also be well to "derive" the inversion formula as the solution of the well-known variational problem whereby the square of the length of the vector difference between a function and a Fourier-represented function is brought to a stationary value.

Turning to the second principle mentioned above, examples in which physical interpretation would have been helpful have already been given (heterodyning, retardation, broadening of spectrum). Further examples are found in the following: while the connection between the convolution integral and Duhamel's formulas are developed, the latter integral is not interpreted physically as the supersposition of the effect of instantaneous heat sources, response to an impulsive blow, etc. Again, the physical interpretation of a Sturm-Liouville problem as the steady state amplitude equation for a generalized string, transmission line, etc. is not given, although the physical significance of the quantization would have emerged as a necessary condition for two traveling waves to give standing waves. The Green's function corresponding to a given boundary condition is deduced, but the interpretation of the discontinuity in its first derivative as a point force, and hence a delta function force density, is not given.

To summarize: This book is an excellent work on the applications of the Laplace transformation. It is believed that it would be more useful to a wider class of readers if it started from the more generally known exponential Fourier integral, which in this edition is granted only a part of one page; if it developed the well-known extension of the terminology of geometry to function-space; and if it interpreted the formalism in physical terms.

University of California
Division of War Research.

GLEN D. CAMP.

Ordinary Differential Equations. By E. L. Ince. Dover Publications, New York, 1944 (reprint of 1927 edition). viii+558 pp. $3.75.

The book under review first appeared in 1927, and is a well-known standard work. Its republication in inexpensive form is a real service to both "*pure*" and "*applied*" mathematicians. A great many topics are covered or touched upon, and the book is useful both as a reference work (particularly, perhaps, with regard to special functions such as Legendre polynomials, Bessel functions, etc.) and as a point of departure for the student who is interested in general theories. There are many references to the literature.

Part I, *Differential Equations in the Real Domain*, opens with two chapters on rather heterogeneous preliminaries and methods of solution. Here one finds the usual tricks taught beginning students, and more. Chapter III takes up deeper ideas, with existence proofs in the familiar Cauchy-Lipschitz and Picard traditions, avoiding assumptions of analyticity. One pleasant feature is the illumination, by examples, of the rôle of the Lipschitz condition in guaranteeing uniqueness. (Here one may also refer to Carathéodory's *Reelle Funktionen.*) Chapter IV shows how the Lie theory of continuous groups is applied to first-order equations in determining integrating factors; this chapter was intended to tie together the existence theory and the special methods. The rest of Part I is occupied with linear equations. Notable chapters deal with equations having analytic (power series) solutions (wherein one finds the special functions that are so useful in the "applications" which gave rise to the subject), boundary-value problems and questions of oscillation. Chapter VII begins with a flaw: the reader is told that a linear equation whose coefficients are merely continuous must have analytic solutions!

Part II, *Differential Equations in the Complex Domain*, is more happily (and quite properly) limited to consideratilon of analytic functions. The opening chapter (XII) takes up Cauchy's existence proof, using majorant functions, which ought to have appeared in Chapter VII. The next two chapters deal with non-linear equations, whose solutions—unlike those of linear equations—may possess essential singularities or branch points whose positions depend on the (arbitrary) constants of integration. Chapter XIV treats second order equations for which these "critical points" are fixed; it includes the Painlevé-Gambier classification of such equations. The rest of Part II is concerned with linear equations and linear systems. The "special functions" appear in both power series and contour integral representations, in chapters which should be especially useful to physicists and others. A chapter on linear systems closes with a statement of G. D. Birkhoff's generalization of the Riemann problem. The last chapter (XXI) contains the most impressive modern material to be found in the book, Hille's extension to the complex domain of the Sturm-Liouville order of ideas. Two appendices follow: an interesting *Historical Note on Formal Methods of Integration*, and an dices follow: an interesting *Historical Note on Formal Methods of Integration*, and an account of the Runge numerical process. The book contains a Bibliography, an Index of Authors, and a General Index.

Northwestern University. A. T. LONSETH.

Engineering Preview. By Grinter, Holmes, Spencer, Oldenburger, Harris, Kloeffler, Faires. Macmillan, New York, 1945

This very interesting little book by a group of leading scientists and engineers presents in summary form a treatment of the basic sciences upon which engineering rests. Considerable attention is also devoted to TECHNICAL DRAWING and the SLIDE RULE. In short, the book is a good guide to what the prospective engineer should

know before he starts to study engineering. It is also a good guide to some of the things a practicing engineer should know.

Accepting Tredgold's definition of ENGINEERING—i. e. "*the art and science of directing the forces of nature to the use and convenience of man*", the authors discuss scientists, engineers and technicians and their place in the modern world. The definition of CULTURE—i. e. "*a deep understanding of man and his environment*", appeals to the reviewer as one of the best statements he has seen. There is considerable discussion of the qualifications of an engineer and a series of aptitude tests is offered. No definition is given of the term "*Engineer*", however, and the reviewer suggests that this might well have been included. The American Society of Civil Engineers defines an ENGINEER (i. e. A full member of the society) as *one who is competent to conceive and design as well as to direct engineering works* and as *one who has had responsible charge of important work for at least five years.* The other major professional engineering societies have similar qualifications for membership. The reviewer would venture the suggestion that *the ability to do independent and original work* is a characteristic of all really great engineers. He is not at all sure however that any type of test can be devised which will give an index to such ability, except the TEST OF PERFORMANCE.

A little more definiteness regarding what is meant by the terms "*engineers*", "*administrators*", and "*supervisors*" would seem desirable in the discussion of the engineer's place and status in society. Dean Grinter estimates that there are 280,000 engineers in the country in 1945. The year book of the American Society of Civil Engineers for 1944 lists the number of full members at 6,059 and total members of all grades as 19,600. The other professional engineering societies have membership lists of about the same order of magnitude. Hence it is evident that the figure of 280,000 includes engineers of all grades and various qualifications. Likewise, the statement that, after twenty years of experience, more than half of all engineers become administrators could be amplified a little, the term administrator, being somewhat general. The reviewer does not know of any comprehensive and reliable surveys on this subject, but his own observation is that about 20% of the graduates of an engineering college will eventually become high officers in fair sized or large corporations. In the opinion of the reviewer, a comprehensive study of this question, including trends, would be very valuable to students, to the engineering profession, and to the public. It is of course outside the subject matter of the text under discussion.

The only part of the book dealing with engineering proper is the first chapter, which is excellent. In the opinion of the reviewer it would have been helpful if this chapter could have been longer since youths are stirred by imagination, and this book is primarily for young men. The remainder of the text, dealing with science, mathematics, and theoretical mechanics is stimulating and excellently written, setting forth the basic principles and illustrating their application. The text should appeal not only to prospective engineers but also to all young people who have an interest in science and technology.

Northwestern University. L. T. WYLY.